Ventricular Wall Stress and Strain

* Anatomy
* Ventricular Pump Function
* Resting Myocardial Mechanics
* Constitutive equations for passive myocardium
* Regional Wall Stress and Strain

Factors Affecting Regional Stress and Strain

<table>
<thead>
<tr>
<th>Secondary and Structural</th>
<th>3D Shape</th>
<th>wall thickness, curvature</th>
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</thead>
<tbody>
<tr>
<td>Tissue Structure</td>
<td>stress-free and unloaded/reference configurations, muscle fiber architecture, connective tissue organization, pericardium, epicardium, and endocardium, coronary vascular anatomy</td>
<td></td>
</tr>
<tr>
<td>Boundary/Initial Conditions</td>
<td>Pressure</td>
<td>filling pressure (preload), arterial pressure (afterload), direct and indirect ventricular interactions</td>
</tr>
<tr>
<td></td>
<td>Constraints</td>
<td>effects of inspiration and expiration, constraints due to the pericardium and its attachments, valves and fibrous valve annuli, chordae tendineae, great vessels, lungs</td>
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<tr>
<td>Material Properties</td>
<td>Resting or Passive</td>
<td>nonlinear finite elasticity, quasilinear viscoelasticity, anisotropy, biphasic poroelasticity</td>
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<td>Active Dynamic</td>
<td>activation sequence, myocardial isotropic and isotonic contractile dynamics, sarcomere length and length history, cellular calcium kinetics and metabolic energy supply</td>
</tr>
</tbody>
</table>

Factors Affected by Regional Stress and Strain

| Direct Factors | regional muscle work, myocardial oxygen demand and energetics, coronary blood flow |
| Electrophysiological Responses | myocardial action potential duration (QT interval), repolarization (T wave morphology), excitability, risk of arrhythmia, growth rate, cardiac looping and septation, value formation, ischemia, arrhythmia, cell dropout, aneurysm rupture, eccentric and concentric hypertrophy, fibrosis, scar formation, transition from hypertrophy to failure, ventricular dilation, infarct expansion, aneurysm formation |
| Vulnerability to Injury | |
| Remodeling, Repair and Adaptation | |
| Progression of Disease | |
Resting Tissue Properties

- Nonlinearity
- Hysteresis
- Creep
- Relaxation
- Preconditioning Behavior
- Strain Softening
- Anisotropy

Uniaxial Resting Mechanics
(Contribution of Collagen)

![Graph showing stress vs. strain with comparison between WT and oim](image)

Passive Biaxial Properties

![Graph showing stress vs. equibiaxial strain](image)
Strain Energy Functions

Transversely Isotropic (Isotropic + Fiber) Exponential

\[ W = 0.21 \left( \lambda F \right)^{-3} + 0.35 \left( \lambda F \right)^{-3} - 1 \]

Transversely Isotropic Exponential

\[ W = 0.6 \left( Q^2 - 1 \right) \]

where, in the dog

\[ Q = 26.7E_{11} + 2.0(Q^2E_{12}E_{22}E_{33}E_{23}) + 14.7(E_{11}E_{22}E_{33}E_{23}) \]

and, in the rat

\[ Q = 9.2E_{11} + 2.0(Q^2E_{12}E_{22}E_{33}E_{23}) + 3.7E_{11}E_{22}E_{33}E_{23} \]

Transversely Isotropic Polynomial

\[ W = 0.36 \left( \lambda F \right)^{-3} + \lambda F \left( \lambda F \right)^{-3} \]

Orthotropic Power Law

\[ W = \frac{3}{32} \lambda F^2 + \frac{3}{30} \lambda F^3 + \frac{3}{31} - 3 \]

Measurement of Myocardial strain

- Radiopaque beads and biplane x-ray
- Piezoelectric crystals
- Video imaging of markers
- Ultrasound
- MRI tagging

Stress Analysis of an Elastic Pressure Vessel

Simplifying assumptions:

- Thick-walled hollow cylindrical tube
- Linear elasticity
- Infinitesimal (Cauchy) strains
- Isotropic material properties
- Simple pressure loading
2-D Linear Elasticity: Compatibility

The infinitesimal Cauchy strain tensor in 2D:

\[ \varepsilon_{xx} = \frac{\partial u}{\partial x}, \quad \varepsilon_{yy} = \frac{\partial v}{\partial y}, \quad \varepsilon_{xy} = \frac{1}{2} \left( \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \right) \]

- Three partial differential
- Two displacement components
- Strains are not independent
- Require an "integrability" condition

This condition is called the Equation of Compatibility

2-D Linear Elasticity: Compatibility

e.g., \( \frac{\partial u}{\partial x} = f(x,y) \) and \( \frac{\partial u}{\partial y} = g(x,y) \).

can only be integrated when \( \frac{\partial f}{\partial y} = \frac{\partial^2 u}{\partial x \partial y} = \frac{\partial g}{\partial x} \)

The "Equation of Compatibility" for strain:

\[ \frac{\partial^2 \varepsilon_{xx}}{\partial y^2} = \frac{\partial^2 \varepsilon_{yy}}{\partial x^2}, \quad \frac{\partial^2 \varepsilon_{xy}}{\partial x \partial y} = \frac{\partial^2 \varepsilon_{yx}}{\partial y \partial x} \]

\[ \Rightarrow \frac{\partial^2 \varepsilon_{xx}}{\partial y^2} + \frac{\partial^2 \varepsilon_{xy}}{\partial x \partial y} = \frac{\partial^2 \varepsilon_{yy}}{\partial x^2} + \frac{\partial^2 \varepsilon_{yx}}{\partial y \partial x} \]

In three dimensions, there are six compatibility equations.

Plane Stress and Plane Strain

Hooke’s Law for isotropic linear elasticity:

\[ T_{ij} = \lambda \varepsilon_{ij} \delta_{ij} + 2\mu \varepsilon_{ij} \]

Plane stress: \( T_{xx}, T_{yy}, T_{xy} = 0 \)
- \( T_{xx}, T_{yy}, \) and \( T_{xy} \) are independent of \( z \).
- \( \varepsilon_{xx} = (T_{xx} - \nu T_{yy}) \)
- \( \varepsilon_{yy} = (T_{yy} - \nu T_{xx}) \)
- \( \varepsilon_{xy} = -\frac{\nu}{E}(T_{xx} + T_{yy}) \)
- \( \varepsilon_{yx} = 0 \)

where: \( \nu = \frac{G}{2(\lambda + \mu)} \); \( E = \frac{\lambda + \mu}{\lambda + 2\mu} \); \( G = \mu = \frac{E}{2(1 + \nu)} \)

Plane strain: no \( z \) displacement
- \( u, v \) are independent of \( z \):
- \( \varepsilon_{xx} = \frac{\partial u}{\partial x}, \quad \varepsilon_{yy} = \frac{\partial v}{\partial y}, \quad \varepsilon_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \)

I.e. \( \varepsilon_{xx} = \varepsilon_{xy} = \varepsilon_{yy} = 0 \)

Inverting Hooke’s Law:

\[ \begin{align*}
\varepsilon_{xx} &= \frac{1}{E} \left[ (1 - \nu) T_{xx} - \nu (1 + \nu) T_{yy} \right] \\
\varepsilon_{yy} &= \frac{1}{E} \left[ (1 - \nu) T_{yy} - \nu (1 + \nu) T_{xx} \right] \\
\varepsilon_{xy} &= \frac{1}{E} \left[ (1 + \nu) T_{xx} - (1 - \nu) T_{yy} \right]
\end{align*} \]
Linear Elasticity: Equations of Motion

Equations of motion $\rho \frac{Dv}{Dt} = \frac{\partial T}{\partial x} + \rho \phi$

No body forces and no inertia $\rightarrow$ Equilibrium equations. In 2-D plane stress or plane strain:

$\frac{\partial T_{xx}}{\partial x} + \frac{\partial T_{xy}}{\partial y} = 0$ \hspace{1cm} (a)

$\frac{\partial T_{xy}}{\partial x} + \frac{\partial T_{yy}}{\partial y} = 0$ \hspace{1cm} (b)

Two equations in three unknowns satisfied by $\phi = \phi(x,y)$ such that:

$\frac{\partial^2 \phi}{\partial x^2} = T_{xx}$ $\frac{\partial^2 \phi}{\partial y^2} = T_{yy}$ $\frac{\partial^2 \phi}{\partial x \partial y} = T_{xy}$

$\phi$ is the "Airy Stress Function"

Example: Plane Strain

Compatibility: $\frac{\partial^2 \varepsilon_{xx}}{\partial y^2} + \frac{\partial^2 \varepsilon_{yy}}{\partial x^2} = \frac{2 \partial^2 \varepsilon_{xy}}{\partial x \partial y}$ Using inverse of Hooke's Law

$\Rightarrow \frac{1}{E} \left[ \frac{1}{E} \left[ \frac{(1-\nu)\varepsilon_{xx}}{\partial y^2} - \nu \varepsilon_{yy} + (1-\nu)\varepsilon_{xy} - \nu \varepsilon_{xx} \right] \right] = \frac{2(1+\nu)}{E} \frac{\partial^2 T_{yy}}{\partial x \partial y}$

Equilibrium equations:

$\frac{\partial (a)}{\partial x} + \frac{\partial (b)}{\partial y} = 0$

$\Rightarrow \frac{\partial^2 T_{xx}}{\partial x^2} + \frac{\partial^2 T_{xy}}{\partial x \partial y} + \frac{\partial^2 T_{yy}}{\partial y^2} = 0$ \hspace{1cm} use this to eliminate $T_{yy}$

$\Rightarrow \frac{(1-\nu)\varepsilon_{xx}}{\partial y^2} + \frac{2\nu T_{xx}}{\partial x} + \frac{\partial^2 \varepsilon_{yy}}{\partial x^2} + \frac{(1-\nu)\varepsilon_{yy} - \nu \varepsilon_{xx}}{\partial x^2} = \frac{\partial T_{xx}}{\partial x} + \frac{\partial^2 T_{yy}}{\partial y^2}$

$\Rightarrow \frac{\partial^2 (T_{xx} + T_{yy})}{\partial x^2} + \frac{\partial^2 (T_{xx} + T_{yy})}{\partial y^2} = 0$

The Biharmonic Equation

In terms of the Airy stress function

$\frac{\partial^2 \left( \frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y} \right)}{\partial x^2} + \frac{\partial^2 \left( \frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y} \right)}{\partial y^2} = \left[ \frac{\partial^4 \phi}{\partial x^4} + 2\frac{\partial^4 \phi}{\partial x^2 \partial y^2} + \frac{\partial^4 \phi}{\partial y^4} \right] = 0$

I.e. the Biharmonic Equation

$\nabla^4 \phi = 0$

Where $\nabla^4 = \left( \frac{\partial^4}{\partial x^4} - 2\frac{\partial^4}{\partial x^2 \partial y^2} + \frac{\partial^4}{\partial y^4} \right)$
Biharmonic equation: Examples

The polynomials $\Phi_1 = \frac{a_1 x^2 + b_1 xy + c_1 y^2}{2}$
and $\Phi_2 = \frac{a_2 x^2 + b_2 x^2 y}{2} + \frac{c_2 xy + d_2 y^2}{6}$
are biharmonic. By adjusting the coefficients $a$, $b$, $c$ and $d$,
a variety of problems on rectangular domains are solved.
From $\Phi_1$: $T_{xx} = c_2$, $T_{yy} = a_2$, $T_{xy} = -b_2$
(constant normal and shear stresses)
From $\Phi_2$: we get a general linear variation in $T_{xx}$, $T_{yy}$, and $T_{xy}$.

Summary of the Stress Function
Approach to Linear Elasticity

• Forward approach to the boundary value problem:
  1. Prescribe displacements in terms of unknown parameters
  2. Derive strains from strain-displacement relation
  3. Derive stresses from constitutive relation
  4. Put stresses in equilibrium equations
  5. Solve equilibrium equations for unknown parameters subject
to boundary conditions
• Stress function approach:
  1. Choose scalar stress function, $\Phi$
  2. Confirm $\Phi$ is biharmonic (compatibility is satisfied)
  3. Derive stresses from second derivatives of $\Phi$
  4. Use boundary conditions to identify unknown parameters of $\Phi$
  5. Derive strains from inverse of constitutive law
  6. Integrate strain-displacement relations to get displacements

2D Problems in Polar Coordinates

Displacements $u_r$, $u_\theta$

Strains $\varepsilon_r = \frac{\partial u}{\partial r}$, $\varepsilon_\theta = \frac{1}{r} \frac{\partial u}{\partial \theta}$, $\gamma_{rr} = \frac{1}{2} \left( \frac{\partial u}{\partial r} + \frac{\partial u}{\partial \theta} - \frac{u}{r} \right)$

Stress Function: $T_r = \frac{1}{r} \frac{\partial \Phi}{\partial r} + \frac{1}{r^2} \frac{\partial^2 \Phi}{\partial \theta^2}$
$T_\theta = \frac{\partial \Phi}{\partial \theta} - \frac{1}{r} \frac{\partial \Phi}{\partial r}$
$\nabla^2 \Phi = \frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial \Phi}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2 \Phi}{\partial \theta^2}$
**Axisymmetric Problems**

\[ \Phi = \Phi(r) \]

\[ \nabla^2 \Phi = \nabla^2 \Phi = \frac{d \Phi}{dr} + \frac{d \Phi}{d r} = \frac{1}{r} \frac{d \Phi}{d r} + \frac{1}{r^2} \frac{d^2 \Phi}{d r^2} = 0 \]

Putting \( r = e^\theta \) gives general solution by making equation a linear ode.

\[ \Phi = A \log r + B r^2 \log r + Cr^2 + D \]

If we now calculate the stresses, use Hooke’s law to get strains and then integrate for displacements, we find that the circumferential displacement \( u_\theta \) is multiple valued unless \( B = 0 \).

Hence \( T_\theta = A \frac{1}{r^2} + 2C \)

\( T_\theta = -A \frac{1}{r^2} + 2C \)

\( T_\theta = 0 \)

**Axisymmetric Problems**

Hence \( T_\theta = \frac{A}{r^2} + 2C \)

\( T_\theta = -\frac{A}{r^2} + 2C \)

\( T_\theta = 0 \)

Boundary Conditions:

\( T_\theta = -p \) at \( r = a \)

\( T_\theta = -q \) at \( r = b \)

\[ \begin{align*}
T_\theta &= -p \left( \frac{b^2}{a^2} \right) \left( 1 - \frac{1}{r^2} \right) \\
T_\theta &= -q \left( \frac{b^2}{a^2} \right) \left( 1 - \frac{1}{r^2} \right)
\end{align*} \]

\( T_\theta = p \left( \frac{b^2}{a^2} \right) \left( 1 - \frac{1}{r^2} \right) \)

\( T_\theta = q \left( \frac{b^2}{a^2} \right) \left( 1 - \frac{1}{r^2} \right) \)

**Example:** \( a = \frac{3}{2} \quad b = \frac{57}{2} \quad (L = 5.6) \) \( \) inner:outer ratio = 2.305

\( p = 1 \text{kPa} \quad q = 0 \rightarrow T_{\theta, a} = 177 \quad T_{\theta, b} = 0.77 \)

<table>
<thead>
<tr>
<th>R</th>
<th>1.5</th>
<th>1.65</th>
<th>1.8</th>
<th>1.95</th>
<th>2.1</th>
<th>2.25</th>
<th>2.4</th>
<th>2.55</th>
<th>2.7</th>
<th>2.85</th>
</tr>
</thead>
<tbody>
<tr>
<td>( T_\theta )</td>
<td>1.77</td>
<td>1.53</td>
<td>1.34</td>
<td>1.20</td>
<td>1.09</td>
<td>1.0</td>
<td>0.82</td>
<td>0.86</td>
<td>0.81</td>
<td>0.77</td>
</tr>
<tr>
<td>( T_{\theta, a} )</td>
<td>-1.0</td>
<td>-0.76</td>
<td>-0.58</td>
<td>-0.44</td>
<td>-0.32</td>
<td>-0.23</td>
<td>-0.16</td>
<td>-0.1</td>
<td>-0.04</td>
<td>0.0</td>
</tr>
</tbody>
</table>

**Axisymmetric Problems**

\( a = 15 \text{ cm} \quad b = 2.85 \text{ cm} \quad (L = 5.6 \text{ cm}) \)

LV pressure \( p = 10 \text{ kPa} = 7.5 \text{ mmHg} \) \( \) External pressure \( q = 0 \)

\[ \text{Stress, } T_{\theta, a} \text{ (kPa)} \]

\[ \begin{align*}
\text{Radius, } r \text{ (cm)} & \quad 1.5 & 1.8 & 2.1 & 2.4 & 2.7 \\
\text{Stress, } T_{\theta, a} \text{ (kPa)} & 1.8 & 1.6 & 1.4 & 1.2 & 1.0 \\
\end{align*} \]

Note \( T_\theta \) and \(-T_\theta\) are parallel. \( T_\theta + T_{\theta, a} = 4C = \frac{2(a^2 p - b^2 q)}{(b^2 - a^2)} \)
Minimizing Stress Gradients

- Residual Stress
- Fiber Angles
- Torsion

Equatorial slice Unloaded state Stress-free state

Ventricular Mechanics: Summary of Key Points

- Collagen contributes to anisotropic resting properties
- Exponential and polynomial strain-energy functions have been used for resting myocardium
- Myocardial strain can be measured invasively and non-invasively
- Linear elasticity analysis of a thick pressure vessel can be solved using the Airy Stress Function
- the Airy stress function satisfies the Biharmonic equation for 2-D Hookean elasticity
- Stress and strains tend to be greatest on the inner surface of thick pressure vessels
- Torsion and residual stress tend to compensate for these gradients in the ventricles to maintain uniform fiber strain